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Propositional Systems and Measurements. III. Quasitensorproducts of Certain Orthomodular Lattices[1]

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Abstract

Continuing our investigations on propositional systems without assumption of the covering law, we introduce a quasi-tensor-product of a complete atomic orthomodular lattice with a complete atomic Boolean lattice. This product has a universal property with respect to postulates on propositional systems of coupled physical systems. We use it to describe measurements on a purely quantal object by a purely classical apparatus and find no nontrivial proposition of the object to be commensurable with its quantal negation. If the object is not purely quantal, the central propositions are commensurable. By this, it is shown directly that useful apparatuses must have a quantal microstructure.

1. *Introduction*

In two preceding papers (1974a,b, henceforth cited as PSM I and PSM II, respectively), we have considered propositional systems for quantal objects for which the covering law is not postulated. Motives for doing so may be found in PSM I. If the covering law is absent, one cannot pass to Hilbert-space representation. Therefore the structure of the propositional systems considered by us is essentially weaker than that of the usual. The structure required by us for a propositional system is that it is always a complete atomic orthomodular lattice.

Since one does not have the linear structure of Hilbert-space for construct-

[1] Dedicated to the 60th birthday of Professor G. Ludwig

ing tensor products, the problem arises how to form the propositional system for a compound object, if the propositional systems of the component objects are given. There are certain minimal requirements on the propositional system of compound objects which we have formulated in lattice-theoretical language as Postulate 5 in PSM II. Assuming that propositional systems exist that fulfil these postulates, we have formulated measurement processes and derived some consequences in PSM II.

Writing the present paper, we found an error in the proof of Lemma 2 of PSM II: The conjunctive normal form we assumed there to hold is not trivial but wrong in general. Fortunately, we found another proof of the lemma, such that the results of PSM II can be maintained. We give this proof in the Appendix.

In the present paper we construct a propositional system T for the compound of a quantal object with a complete atomic orthomodular lattice L as propositional system and a "purely" classical apparatus with a complete atomic Boolean lattice B as propositional system.

We show T to possess a universal property with respect to Postulate 5. T can be embedded into any other propositional system that fulfils Postulate 5 for given L and B. We call T the quasi-tensor-product $B \otimes L$ of B and L. We have called it a quasi-tensor-product because it is not a tensor product in the categorical sense but has a similar universal property. Until now, we have not succeeded in constructing such a propositional system in the case where B is an arbitrary complete atomic orthomodular lattice.

For a quantal object with propositional system L and an apparatus with propositional system B we then investigate measurement processes. We find that the purely classical nature of the apparatus which we have to assume imposes severe restrictions on the measurement possibilities: if the center of L is trivial, then no proposition unequal to $\varnothing$ or I is commensurable with its quantal negation. In the language of Ludwig (1970) this result can be stated as follows: There are no decision effects. Therefore we have formally shown that an apparatus defining decision effects, or, measuring propositions ideally, must have a nonclassical microstructure behind the classical macrobehavior which enters essentially into the measurement process. If L has a nontrivial center Z, we show that the propositions in Z are commensurable and can be measured together ideally by a "purely" classical apparatus.

In Section 2 we review shortly the contents of PSM II that we will use in the present paper. The construction of the quasi-tensor-product is given in Section 3. In Section 4, measurement processes are considered.

2. *Summary of Results of PSM II*

Any measurement on a physical object by some physical apparatus presupposes some coupling of both physical systems defining the compound system. Let the propositional systems L of the object, and B of the apparatus, be given. What has to be required for the propositional system of the compound system? In PSM II we have written the following postulate.[2]

Postulate 5.1. Let T denote the propositional system of a compound system; then there are embeddings

$$\theta_B : B \to T \qquad \text{and } \theta_L : L \to T$$

with the property $\theta_B(\varnothing) = \varnothing$, $\theta_L(\varnothing) = \varnothing$, $\varnothing$ denoting the absurd proposition in any of the three propositional systems.

Postulate 5.2. Let $b \in B$ and $a \in L$. Then $\theta_B(b)$ is compatible with $\theta_L(a)$, for short $\theta_B(b) \leftrightarrow \theta_L(a)$. If both b and a are nontrivial and nonabsurd propositions, then neither $\theta_B(b) \leqslant \theta_L(a)$ nor $\theta_L(a) \leqslant \theta_B(b)$ holds.

Postulate 5.3. Let there be given $B_B \subseteq B$ and $B_L \subseteq L$, B_B and B_L being maximal boolean sublattices in B and L, respectively; then the completion by cuts of the Boolean sublattice generated by $\theta_B(B_B) \cup \theta_L(B_L)$ is a maximal Boolean sublattice of T.

If such a propositional system T exists, measurement processes have been described in PSM II as follows. Let $b_{\text{in}} \in B$ denote the infimum of all propositions imposed as true by the preparation of the apparatus when the measurement process begins, $b_{\text{out}} \in B$ the proposition that is finally to be observed and concludes the measurement process. Let U denote the lattice automorphism of T that describes the temporal development of the compound system from the beginning of the measurement process up to the observation of $\theta_B(b_{\text{out}})$. $\theta_B(b_{\text{out}})$ will then be true at the observation time if and only if $m := U^{-1}(\theta_B(b_{\text{out}}))$ is true at the beginning. Let

(j) $$a_1 := \text{l.u.b. } \{a \in L \mid \theta_L(a) \wedge \theta_B(b_{\text{in}}) \leqslant m\}$$

(jj) $$a_0 := \text{l.u.b. } \{a \in L \mid \theta_L(a) \wedge \theta_B(b_{\text{in}}) \leqslant \psi m\}$$

where ψ denotes the orthocomplementation in T. Imagine now that a long series of experiments has been carried out under the same conditions and b_{out} has occurred in every single case. Then, by the very meaning of the order relation in T, we have

$$\theta_L(a_1) \wedge \theta_B(b_{\text{in}}) \leqslant m$$

and conclude a_1 to hold true for the ensemble of object systems. If b_{out} has not occurred in any single case, we have

$$\theta_L(a_0) \wedge \theta_B(b_{\text{in}}) \leqslant \psi m$$

In this case a_0 is concluded to be true for the ensemble of object systems.

Different propositions in L, which may be detected by several different

[2] Since we do not consider mainly questions in this paper, but only propositions, we do not use the bracket notation as in PSM I or PSM II.

outcomes possible at the end of one and the same measurement process, are called coexistent or, by the very meaning of the word, commensurable. For example, a_1 is commensurable to a_0.

In general, we do not have $a_0 = \varphi a_1$, but rather $a_0 \leqslant \varphi a_1$, where φ denotes the orthocomplementation in both B and L. In Theorem 3 of PSM II we have proved that $a_0 = \varphi a_1$ holds true, if and only if the three relations

(i) $\theta_B(b_{\text{in}}) \leftrightarrow m$

(ii) $\theta_L(a_1) \wedge \theta_B(b_{\text{in}}) = m \wedge \theta_B(b_{\text{in}})$

(iii) $\theta_L(a_0) \wedge \theta_B(b_{\text{in}}) = \psi m \wedge \theta_B(b_{\text{in}})$

hold. It is generally assumed in quantum theory that any proposition a in L is commensurable with its quantal negation φa. Hence the existence of at least one apparatus is assumed such that $a_1 = a$ and $a_0 = \varphi a$.

Another general assumption is that compatibility is equivalent to commensurability. We have shown in Theorem 4 of PSM II that commensurability implies compatibility in our scheme, but we could not show formally that the converse also holds.

In the sequel, an isomorphism or monomorphism is always understood to be a bijective or injective lattice morphism, respectively, which is natural with respect to the whole structure of propositional systems. We call any map that preserves the lattice operations $\wedge$ and $\vee$ a lattice morphism. By "embedding" we mean a monomorphism that preserves arbitrary joins, and, since naturality holds with respect to the orthocomplementation, it preserves arbitrary meets too.

3. *Construction of the Quasi-Tensor-Product*

Our propositional systems are complete atomic orthomodular lattices. The covering law is not assumed to hold. Hence there is no known procedure to construct a propositional system T that verifies Postulate 5 with respect to the given propositional systems B and L. So it is not clear whether such a T exists in general.

We now construct such a T in the special case that one of the given propositional systems, say B, is Boolean.

Denote by $\mathscr{A}(N)$ the subset of atoms of some lattice N. Consider the set $\mathscr{M}(B, L)$ of all mappings

$$f: \mathscr{A}(B) \to L$$

For $f, g \in \mathscr{M}(B, L)$ we write $f \leqslant g$ if and only if $f(e) \leqslant g(e)$ for every $e \in \mathscr{A}(B)$. Obviously, by this, $\mathscr{M}(B, L)$ is endowed with an antisymmetric partial ordering. Let φ denote the orthocomplementation in L. We define

$$\psi : \mathscr{M}(B, L) \to \mathscr{M}(B, L)$$

$$f \mapsto \varphi \circ f$$

Proposition 3.1. If L is a complete atomic orthomodular lattice and B is a complete atomic Boolean lattice, then $\mathscr{M}(B, L)$ is a complete atomic orthomodular lattice. The orthocomplementation on $\mathscr{M}(B, L)$ is given by ψ.

Proof. Infima and suprema on arbitrary families $\{f_k\}_{k\in\mathscr{K}} \subseteq \mathscr{M}(B, L)$ are obviously given by

$$\bigwedge_{k\in\mathscr{K}} f_k : \mathscr{A}(B) \to L$$

$$e \mapsto \bigwedge_{k\in\mathscr{K}} f_k(e)$$

and

$$\bigvee_{k\in\mathscr{K}} f_k : \mathscr{A}(B) \to L$$

$$e \mapsto \bigvee_{k\in\mathscr{K}} f_k(e)$$

Since L is complete, so is $\mathscr{M}(B, L)$. The lowest element of $\mathscr{M}(B, L)$ is given by the mapping $\mathscr{A}(B) \to \{\varnothing\}$, where $\varnothing$ denotes the lowest element in L. Analogously, the greatest element of $\mathscr{M}(B, L)$ is given by the mapping $\mathscr{A}(B) \to \{I\}$, where I denotes the greatest element in L. Since no confusion arises, we denote these mappings in $\mathscr{M}(B, L)$ also by $\varnothing$, and I, respectively.

The relations $\psi(f) \wedge f = \varnothing$, $\psi(f) \vee f = I$, $\psi(g) \leqslant \psi(f)$ if $g \geqslant f$, and $\psi \circ \psi(f) = f$ are easy to check for $f, g \in \mathscr{M}(B, L)$ by inserting an arbitrary $e \in \mathscr{A}(B)$ into the respective functions and finding the relations true in L. Analogously, one shows that $f \leqslant g$ implies $g = f \vee [\psi(f) \wedge g]$. Hence $\mathscr{M}(B, L)$ is orthomodular.

For any pair $(e_0, \epsilon) \in \mathscr{A}(B) \times \mathscr{A}(L)$ define

$$f_{e_0,\epsilon} : \mathscr{A}(B) \to L$$

$$e \mapsto \begin{cases} \epsilon & \text{if } e = e_0 \\ \varnothing & \text{otherwise} \end{cases}$$

$f_{e_0,\epsilon}$ is obviously an atom in $\mathscr{M}(B, L)$. Now let $f \in \mathscr{M}(B, L)$, $f \neq \varnothing$, say $f(\tilde{e}) \neq \varnothing$. Since L is atomic, there is an $\tilde{\epsilon} \in \mathscr{A}(L)$ such that $\tilde{\epsilon} \leqslant f(\tilde{e})$. Hence $f_{\tilde{e},\tilde{\epsilon}} \leqslant f$. This proves $\mathscr{M}(B, L)$ to be atomic. □

Define now

$$\tau : L \to \mathscr{M}(B, L)$$

by $\tau(a)(e) = a$, $a \in L$, $e \in \mathscr{A}(B)$; i.e., $\tau(a)$ maps any $e \in \mathscr{A}(B)$ into $a \in L$. Moreover, define

$$\sigma : B \to \mathscr{M}(B, L)$$

such that for $b \in B$, $\sigma(b)(e) = I$ if $e \leqslant b$, and $\sigma(b)(e) = \varnothing$ otherwise. Then we have the following.

Proposition 3.2. τ and σ are embeddings of L and B into $\mathscr{M}(B, L)$, respectively, which fulfil postulates 5.1-5.3.

Proof. The embedding property of τ is obvious. Since B is assumed to be complete, atomic, and Boolean, B is isomorphic to the lattice of subsets of $\mathscr{A}(B)$ in the natural way, which is isomorphic to the lattice of the characteristical functions of $\mathscr{A}(B)$. But $\sigma(b), b \in B$, is nothing but the characteristic function for the set of atoms $\{e \in \mathscr{A}(B) | e \leqslant b\}$, if the values $\{\varnothing, I\}$ are replaced by $\{0, 1\}$. In this way one easily infers that σ is an embedding. Since $\tau(\varnothing) = \varnothing$, $\tau(I) = I$, $\sigma(\varnothing) = \varnothing$, and $\sigma(I) = I$, Postulate 5.1 holds.

We next show postulate 5.2 to hold. For the first part, we note that $\mathscr{M}(B, L)$ is orthomodular. Hence we only have to prove the equation $[\tau(a) \wedge \sigma(b)] \vee [\tau(a) \wedge \psi \circ \sigma(b)] = \tau(a), a \in L, b \in B,$ since then $\tau(a) \leftrightarrow \sigma(b)$ by a well-known theorem (Piron, 1964). This is easily done by evaluating the respective functions for arbitrary $e \in \mathscr{A}(B)$, by recalling that $\sigma(b)$ takes only values in $\{\varnothing, I\} \subseteq L$, and that $\sigma(b)(e) = \varnothing$ is equivalent to $\varphi \circ \sigma(b)(e) = I$. In order to show the second part, assume $\tau(a) \leqslant \sigma(b)$ to hold. Then $\tau(a)(e) = a \leqslant \sigma(b)(e)$ for every $e \in \mathscr{A}(B)$. Hence $a \neq \varnothing$ implies $\sigma(b)(e) = I$ for every $e \in \mathscr{A}(B)$, i.e., $b = I$. On the other hand $b \neq I$ implies $a = \varnothing$. Assumption of $\sigma(b) \leqslant \tau(a)$ leads to similar results. Hence Postulate 5.2 has been proven.

We now have to show Postulate 5.3 to hold. This will be done by several steps. For an arbitrary maximal Boolean sublattice B_L of L let M denote the Boolean sublattice generated by $\sigma(B) \cup \tau(B_L)$ and let $\bar{M}$ denote the infimum of all complete Boolean sublattices of $\mathscr{M}(B, L)$ that contain M. Moreover let N denote the set of functions in $\mathscr{M}(B, L)$ with range contained in B_L.

We show $N = \bar{M}$. Let $f \in N$, $b \in B$, and $c \in B_L$, then $f \leftrightarrow \sigma(b)$ and $f \leftrightarrow \tau(c)$, such that we have the identity

$$f = f \wedge \left[\bigvee_{\tilde{e} \in \mathscr{A}(B)} \sigma(\tilde{e}) \right] = \bigvee_{\tilde{e} \in \mathscr{A}(B)} [f \wedge \sigma(\tilde{e})]$$

$$= \bigvee_{\tilde{e} \in \mathscr{A}(B)} [\tau(f(\tilde{e})) \wedge \sigma(\tilde{e})]$$

Since $\tau(f(\tilde{e})) \wedge \sigma(\tilde{e}) \in M$, $f \in \bar{M}$. Hence $N \subseteq \bar{M}$. Equality in the latter relation is implied if N is a complete Boolean lattice that contains $\sigma(B) \cup \tau(B_L)$. The range of functions in $\sigma(B)$ is in $\{\varnothing, I\}$, and the range of functions in $\tau(B_L)$ is in B_L; thus $\sigma(B) \cup \tau(B_L) \subseteq N$. Since B_L is maximal Boolean in the complete lattice L, B_L is complete, and, in consequence, N is closed with respect to arbitrary meets, joins, and the orthocomplementation ψ. Moreover, N is Boolean.

We show N to be maximal Boolean in $\mathscr{M}(B, L)$. Let $g \in \mathscr{M}(B, L)$, $g \leftrightarrow f$ for any $f \in N$; then $g \leftrightarrow \sigma(b)$ and $g \leftrightarrow \tau(c)$ for any $b \in B$ and any $c \in B_L$. Hence, by the same argument as above showing us that $N \subseteq \bar{M}$, we have $g \in \bar{M} = N$.

In order to show that N equals the completion by cuts of M we use a result of McLaren (1964, Theorem 2.5). A subset W of a partially ordered set S is called join dense, if any element $s \in S$ is a finite or arbitrary infinite join of elements of W. McLaren has shown that the completion by cuts of an orthocomplemented lattice is isomorphic to the completion by cuts of any join dense subset. Now $h \in \bar{M}$ implies $h \leftrightarrow \sigma(b)$ and $h \leftrightarrow \tau(c)$ for any $b \in B$ and any $c \in B_L$, such that h is the union of $\{\tau(h(\tilde{e})) \wedge \sigma(\tilde{e}) \mid \tilde{e} \in \mathscr{A}(B)\}$, which is a subset of M. Hence, the completion by cuts of $\bar{M}$ is isomorphic to the completion by cuts of M. On the other hand, the completion by cuts of the complete lattice $\bar{M}$ coincides with $\bar{M}$. So, for the completion by cuts $\hat{M}$ of M, we have $\hat{M} = \bar{M} = N$. □

Corollary 3.3. (1) Let Z denote the center of L. Then the completion of $\tau(Z) \cup \sigma(B)$ is in the center of $\mathscr{M}(B, L)$. (2) In case Z is trivial, $\sigma(B)$ coincides with the center of $\mathscr{M}(B, L)$.

Proof. Statement (1) holds true if $\tau(Z) \cup \sigma(B)$ is in the center of $\mathscr{M}(B, L)$. Now, $g \in \mathscr{M}(B, L)$ is in the center if $f(e) = [g(e) \wedge f(e)] \vee [\varphi g(e) \wedge f(e)]$ holds for any $f \in \mathscr{M}(B, L)$ and any $e \in \mathscr{A}(B)$. For $g \in \sigma(B)$ we have $g(e) \in \{\varnothing, I\}$, hence the equation is trivial. For $g \in \tau(Z)$, i.e., $g(e) = a$, $a \in Z$, for any $e \in \mathscr{A}(B)$, the equation also is trivial since $f(e) \in L$. Thus statement (1) is proved.

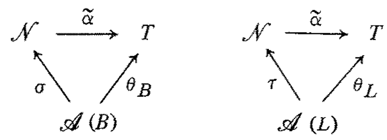
In order to prove statement (2) assume $f \in Z_{\mathscr{M}}$, $Z_{\mathscr{M}}$ denoting the center of $\mathscr{M}(B, L)$. Then $f \leftrightarrow g$ for any $g \in \mathscr{M}(B, L)$ and, especially, $f(e) \leftrightarrow g(e)$ in L for any $e \in \mathscr{A}(B)$. Since Z is supposed to be trivial, $f(e) \in \{\varnothing, I\}$. Hence $f = \sigma(a)$, where $a := \bigvee \{e \mid f(e) = I\}$ in B. So $Z_{\mathscr{M}} \subseteq \sigma(B)$. But $\sigma(B) \subseteq Z_{\mathscr{M}}$ by statement (1). Hence $\sigma(B) = Z_{\mathscr{M}}$. □

We are now going to show that $\mathscr{M}(B, L)$ has a universal property with respect to Postulates 5.1–5.3: $\mathscr{M}(B, L)$ can be embedded into any lattice T fulfilling Postulates 5.1–5.3 with respect to B and L.

Lemma 3.4. Let T be any complete atomic orthomodular lattice such that embeddings

$$\theta_B : B \to T \qquad \theta_L : L \to T$$

exist, which verify Postulates 5.1–5.3. Let $\mathscr{N} := \sigma(\mathscr{A}(B)) \cup \tau(\mathscr{A}(L))$. Then any mapping $\tilde{\alpha}: \mathscr{N} \to T$, for which the diagrams



are commutative, can be extended uniquely to a lattice monomorphism $\alpha: \mathscr{M}(B, L) \to T$, which preserves arbitrary joins.

Proof. The atoms of $\mathscr{M}(B, L)$ are obviously the elements of the form $f_{e,\epsilon} := \sigma(e) \wedge \tau(\epsilon)$, (e, ϵ) varying in $\mathscr{A}(B) \times \mathscr{A}(L)$. We have $f_{e,\epsilon}(\tilde{e}) = I \wedge \epsilon = \epsilon$ for $\tilde{e} = e$, $f_{e,\epsilon}(\tilde{e}) = \varnothing$ otherwise. Since $\mathscr{M}(B, L)$ is atomic, any $f \in \mathscr{M}(B, L)$ can be represented by

$$f = \bigvee_{(i, k) \in \mathscr{I}} f_{e_i, \epsilon_k}$$

where $\mathscr{I}$ is some suitable indexing set and $(e_i, \epsilon_k) \in \mathscr{A}(B) \times \mathscr{A}(L)$, $(i, k) \in \mathscr{I}$. Let $\mathscr{I}_1 := \{i \mid (i, k) \in \mathscr{I}, k \text{ suitable}\}$ and for any $i \in \mathscr{I}_1$ let $\mathscr{I}_2^{(i)} = \{k \mid (i, k) \in \mathscr{I}, i \text{ fixed}\}$. The indexing sets can always be chosen in such a manner that $e_i \neq e_j$ whenever $i \neq j$ $(i, j \in \mathscr{I}_1)$. Thus the representation of f can be written as

$$f = \bigvee_{i \in \mathscr{I}_1} \left(\bigvee_{k \in \mathscr{I}_2^{(i)}} f_{e_i, \epsilon_k} \right)$$

In the following, all indexing sets for elements $f \in \mathscr{M}(B, L)$ are assumed to be of that kind.

We will show that $\alpha(f)$ is given by

$$\alpha(f) = \bigvee_{(i, k) \in \mathscr{I}} [\tilde{\alpha} \circ \sigma(e_i) \wedge \tilde{\alpha} \circ \tau(\epsilon_k)]$$

when $f \in \mathscr{M}(B, L)$ is represented by

$$f = \bigvee_{(i, k) \in \mathscr{I}} [\sigma(e_i) \wedge \tau(\epsilon_k)]$$

We have to show that $\alpha(f)$ is defined independently of the particular representation and that α extends $\tilde{\alpha}$.

We prove that α, if it is well defined, is an extension of $\tilde{\alpha}$. We have

$$\alpha \circ \tau(\epsilon) = \bigvee_{e \in \mathscr{A}(B)} [\theta_B(e) \wedge \theta_L(\epsilon)]$$

$$= \left[\bigvee_{e \in \mathscr{A}(B)} \theta_B(e) \right] \wedge \theta_L(\epsilon) = \theta_L(\epsilon) = \tilde{\alpha} \circ \tau(\epsilon)$$

and, if $\{\epsilon_k\}_{k \in \mathscr{K}} \subseteq \mathscr{A}(L)$ is such a family that

$$\bigvee_{k \in \mathscr{K}} \epsilon_k = I$$

$$\alpha \circ \sigma(e) = \bigvee_{k \in \mathscr{K}} [\theta_B(e) \wedge \theta_L(\epsilon_k)] = \theta_B(e) \wedge \left[\bigvee_{k \in \mathscr{K}} \theta_L(\epsilon_k) \right] = \theta_B(e) = \tilde{\alpha} \circ \sigma(e)$$

The representation of $\tau(\epsilon)$ by atoms is unique, hence α is well defined on $\tau(\mathscr{A}(L))$. The representations of $\sigma(e)$ by atoms are easily seen all to be of the form used above, hence α is well defined on $\sigma(\mathscr{A}(B))$.

We now prove that α is also well defined on any other element of $\mathscr{M}(B, L)$. It is obviously well defined on the atoms of $\mathscr{M}(B, L)$, i.e., the elements represented uniquely by $\sigma(e) \wedge \tau(\epsilon)$, $(e, \epsilon) \in \mathscr{A}(B) \times \mathscr{A}(L)$. In the general case, assume

$$f = \bigvee_{i \in \mathscr{I}_1} \left(\bigvee_{k \in \mathscr{I}_2^{(i)}} f_{e_i, \epsilon_k} \right) = \bigvee_{j \in \mathscr{K}_1} \left(\bigvee_{m \in \mathscr{K}_2^{(j)}} f_{\tilde{e}_j, \tilde{\epsilon}_m} \right)$$

Now $f(e) \neq \varnothing$ if and only if there is an $i \in \mathscr{I}_1$ such that $e = e_i$, and, analogeously, there is a $j \in \mathscr{K}_1$, too, such that $e = \tilde{e}_j$. Since we consider only indexing sets of the kind that $i \mapsto e_i$ and $j \mapsto \tilde{e}_j$ are injective, we conclude that there is a bijection $\kappa : \mathscr{I}_1 \to \mathscr{K}_1$ such that $e_i = \tilde{e}_{\kappa(i)}$ for all $i \in \mathscr{I}_1$. Recalling the property of functions $f_{e,\epsilon}$, we have

$$f(e) = \bigvee_{k \in \mathscr{I}_2^{(i)}} \epsilon_k = \bigvee_{m \in \mathscr{K}_2^{\kappa(i)}} \tilde{\epsilon}_m, \qquad e = e_i$$

Hence, using $f_{e_i, \epsilon_k} = \sigma(e_i) \wedge \tau(\epsilon_k)$ and Postulate 5.2, we deduce

$$\begin{aligned}\bigvee_{k \in \mathscr{I}_2^{(i)}} [\tilde{\alpha} \circ \sigma(e_i) \wedge \tilde{\alpha} \circ \tau(\epsilon_k)] &= \theta_B(e_i) \wedge \left[\bigvee_{k \in \mathscr{I}_2^{(i)}} \theta_L(\epsilon_k) \right] \\ &= \theta_B(e_i) \wedge \left[\bigvee_{m \in \mathscr{K}_2^{\kappa(i)}} \theta_L(\tilde{\epsilon}_m) \right] \\ &= \bigvee_{m \in \mathscr{K}_2^{\kappa(i)}} [\tilde{\alpha} \circ \sigma(\tilde{e}_{\kappa(i)}) \wedge \tilde{\alpha} \circ \tau(\tilde{\epsilon}_m)]\end{aligned}$$

Forming the supremum over $\mathscr{I}_1$, we have

$$\bigvee_{i \in \mathscr{I}_1} \left\{ \bigvee_{k \in \mathscr{I}_2^{(i)}} [\tilde{\alpha} \circ \sigma(e_i) \wedge \tilde{\alpha} \circ \tau(\epsilon_k)] \right\} = \bigvee_{j \in \mathscr{K}_1} \bigvee_{m \in \mathscr{K}_2^{(j)}} [\tilde{\alpha} \circ \sigma(\tilde{e}_j) \wedge \tilde{\alpha} \circ \tau(\tilde{\epsilon}_m)]$$

which is the desired result that $\alpha(f)$ does not depend on the particular representation of f.

Let there now be given an arbitrary family $\{g_h\}_{h \in \mathscr{H}} \subseteq \mathscr{M}(B, L)$. The relation

$$\alpha\left(\bigvee_{h \in \mathscr{H}} g_h \right) = \bigvee_{h \in \mathscr{H}} \alpha(g_h)$$

follows directly from the construction of α. Somewhat more involved is the proof of

$$\alpha\left(\bigwedge_{h \in \mathscr{H}} g_h \right) = \bigwedge_{h \in \mathscr{H}} \alpha(g_h)$$

We have, using Corollary 3.3,

$$\begin{aligned}\bigwedge_{h \in \mathscr{H}} g_h &= \left(\bigwedge_{h \in \mathscr{H}} g_h \right) \wedge \left[\bigvee_{e \in \mathscr{A}(B)} \sigma(e) \right] = \bigvee_{e \in \mathscr{A}(B)} \left[\left(\bigwedge_{h \in \mathscr{H}} g_h \right) \wedge \sigma(e) \right] \\ &= \bigvee_{e \in \mathscr{A}(B)} \left\{ \bigwedge_{h \in \mathscr{H}} [g_h \wedge \sigma(e)] \right\}\end{aligned}$$

Let

$$g_h = \bigvee_{i_h \in \mathscr{I}_{h,1}} \left(\bigvee_{k_h \in \mathscr{I}_{h,2}^{(i_h)}} f_{e_{i_h}^{(h)}, \epsilon_{k_h}^{(h)}} \right)$$

be a representation of g_h by a supremum of atoms. Define

$$\{e_k\}_{k \in K} := \bigcap_{(h, i_h) \in \mathscr{L}} e_{i_h}^{(h)}$$

where

$$\mathscr{L} = \bigcup_{h \in \mathscr{H}} (\{h\} \times \mathscr{I}_{h,1})$$

If $e \neq e_k$ for all $k \in \mathscr{K}$ then there is an $h \in \mathscr{H}$ such that $g_h \wedge \sigma(e) = \varnothing$. This follows easily, recalling that

$$g_h = \bigvee_{i_h \in \mathscr{I}_{h,1}} \left\{ \sigma(e_{i_h}^{(h)}) \wedge \left[\bigvee_{k_h \in \mathscr{I}_{h,2}}^{(i_h)} \tau(\epsilon_{k_h}^{(h)}) \right] \right\}$$

As a consequence we have $\bigwedge_{h \in \mathscr{H}} [g_h \wedge \sigma(e)] = \varnothing$, hence

$$\bigwedge_{h \in \mathscr{H}} g_h = \bigvee_{k \in \mathscr{K}} \left\{ \bigwedge_{h \in \mathscr{H}} [g_h \wedge \sigma(e_k)] \right\}$$

Let now denote $\iota_h : \mathscr{K} \to \mathscr{I}_{h,1}$ the injections defined through $e_k = e_{\iota_h(k)}^{(h)}$, $k \in \mathscr{K}$, $h \in \mathscr{H}$. Then

$$g_h \wedge \sigma(e_k) = \sigma(e_k) \wedge \left[\bigvee_{n_h \in \mathscr{I}_{h,2}}^{[\iota_h(k)]} \tau(\epsilon_{n_h}^{(h)}) \right]$$

So we have

$$\bigwedge_{h \in \mathscr{H}} g_h = \bigvee_{k \in \mathscr{K}} \left\{ \bigwedge_{h \in \mathscr{H}} \left[\sigma(e_k) \wedge \left(\bigvee_{n_h \in \mathscr{I}_{h,2}}^{[\iota_h(k)]} \tau(\epsilon_{n_h}^{(h)}) \right) \right] \right\}$$

$$= \bigvee_{k \in \mathscr{K}} \left\{ \sigma(e_k) \wedge \left[\bigwedge_{h \in \mathscr{H}} \left(\bigvee_{n_h \in \mathscr{I}_{h,2}}^{[\iota_h(k)]} \tau(\epsilon_{n_h}^{(h)}) \right) \right] \right\}$$

$$= \bigvee_{k \in \mathscr{K}} \bigvee_{m_k \in \mathscr{K}^{(k)}} [\sigma(e_k) \wedge \tau(\tilde{\epsilon}_{m_k}^{(k)})]$$

In the latter step we have put

$$\bigwedge_{h \in \mathscr{H}} \left[\bigvee_{n_h \in \mathscr{I}_{h,2}}^{[\iota_h(k)]} \tau(\epsilon_{n_h}^{(h)}) \right] = \bigvee_{m_k \in \mathscr{K}^{(k)}} \tau(\tilde{\epsilon}_{m_k}^{(k)})$$

$\tilde{\epsilon}_{m_k}^{(k)} \in \mathscr{A}(L)$, which is possible since L is atomic, and again used Corollary 3.3. Now, by definition of α, we have

$$\alpha \left(\bigwedge_{h \in \mathscr{H}} g_h \right) = \bigvee_{k \in \mathscr{K}} \bigvee_{m_k \in \mathscr{K}^{(k)}} [\theta_B(e_k) \wedge \theta_L(\tilde{\epsilon}_{m_k}^{(k)})]$$

On the other hand, we have

$$\bigwedge_{h\in\mathscr{H}} \alpha(g_h) = \bigwedge_{h\in\mathscr{H}} \left\{ \bigvee_{i_h\in\mathscr{I}_{h,1}} \left[\theta_B(e_{i_h}^{(h)}) \wedge \left(\bigvee_{k_h\in\mathscr{I}_{h,2}^{(i_h)}} \theta_L(\epsilon_{k_h}^{(h)}) \right) \right] \right\}$$

Since the arguments just used to obtain the final expression for $\wedge_{h\in\mathscr{H}} g_h$ in $\mathscr{M}(B, L)$ also apply for computing $\wedge_{h\in\mathscr{H}} \alpha(g_h)$ in T, we find

$$\bigwedge_{h\in\mathscr{H}} \alpha(g_h) = \bigvee_{k\in\mathscr{K}} \left\{ \theta_B(e_k) \wedge \left[\bigwedge_{h\in\mathscr{H}} \left(\bigvee_{n_h\in\mathscr{I}_{h_2}^{[\iota_h(k)]}} \theta_L(\epsilon_{n_h}^{(h)}) \right) \right] \right\}$$

$$= \bigvee_{k\in\mathscr{K}} \bigvee_{m_k\in\mathscr{K}^{(k)}} [\theta_B(e_k) \wedge \theta_L(\tilde{\epsilon}_{m_k}^{(k)})]$$

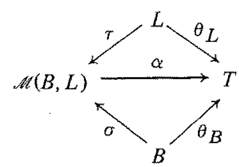
which coincides with the expression we derived for $\alpha(\wedge_{h\in\mathscr{H}} g_h)$.

In order to show $\alpha \circ \psi = \psi \circ \alpha$, where ψ denotes the orthocomplementation in both, $\mathscr{M}(B, L)$ and T, we can restrict our consideration to atoms, since by naturality of α with respect to arbitrary meets and joins the general case is then implied. Let $e \in \mathscr{A}(B)$, $\epsilon \in \mathscr{A}(L)$, then $\alpha \circ \psi\, (\sigma(e) \wedge \tau(\epsilon)) = \alpha(\psi \circ \sigma(e) \vee \psi \circ \tau(\epsilon)) = \alpha(\sigma \circ \varphi(e) \vee \tau \circ \varphi(\epsilon)) = \theta_B \circ \varphi(e) \vee \theta_L \circ \varphi(\epsilon) = \psi \circ \theta_B(e) \vee \psi \circ \theta_L(\epsilon) = \psi(\theta_B(e) \wedge \theta_L(\epsilon)) = \psi \circ \alpha(\sigma(e) \wedge \tau(\epsilon))$, where φ denotes the orthocomplementation in both B and L.

Since $\mathscr{M}(B, L)$ is atomic orthomodular and $\psi \circ \alpha = \alpha \circ \psi$, α is a monomorphism if the restriction on atoms is injective. Let $e, \tilde{e} \in \mathscr{A}(B)$; $\epsilon, \tilde{\epsilon} \in \mathscr{A}(L)$. Then $\sigma(e) \wedge \tau(\epsilon) \neq \sigma(\tilde{e}) \wedge \tau(\tilde{\epsilon})$ implies that one of the relations $e \wedge \tilde{e} = \varnothing$ or $\epsilon \wedge \tilde{\epsilon} = \varnothing$ holds. Hence, $\theta_B(e) \wedge \theta_B(\tilde{e}) = \varnothing$ or $\theta_L(\epsilon) \wedge \theta_L(\tilde{\epsilon}) = \varnothing$, which in turn implies $\theta_B(e) \wedge \theta_L(\epsilon) \neq \theta_B(\tilde{e}) \wedge \theta_L(\tilde{\epsilon})$ since otherwise both sides could be shown to equal zero. The latter cannot happen because of Postulate 5 holding for θ_B, θ_L. Since $\alpha(\sigma(e) \wedge \tau(\epsilon)) = \theta_B(e) \wedge \theta_L(\epsilon)$, the statement is proven. The uniqueness of α is obvious by the definition of α. This concludes the proof of Lemma 3.4. □

We now state the proposed universal property of the lattice $\mathscr{M}(B, L)$.

Theorem 3.5. Let T be any complete atomic orthomodular lattice, and $\theta_B : B \to T$, $\theta_L : L \to T$ be embeddings such that Postulates 5.1–5.3 are fulfilled. Then there is exactly one monomorphism $\alpha : \mathscr{M}(B, L) \to T$ such that the diagram



is commutative and α is natural with respect to the orthocomplementation and arbitrary joins.

Proof. Since $\tilde{\alpha}$: $\mathscr{N} \to T$, defined by $\tilde{\alpha} \circ \sigma(e) := \theta_B(e)$, $e \in \mathscr{A}(B)$, and $\tilde{\alpha} \circ \tau(\epsilon) := \theta_L(\epsilon)$, $\epsilon \in \mathscr{A}(L)$,, is the only mapping that makes the diagrams of Lemma 3.4 commutative, the theorem is a direct consequence of Lemma 3.4.□

Definition 3.6. Given B and L as above, $\mathscr{M}(B, L)$ is called "*quasi-tensor-product of a complete atomic Boolean lattice B with the complete atomic orthomodular lattice L*," and will be denoted by $B \otimes L$.

$B \otimes L$ can be understood as the unique class of isomorphic complete atomic orthomodular lattices that is minimal in the sense of Theorem 3.5. In the following section we apply this to measuring processes. Before doing so, we mention some statements without proof for the special case where L is also Boolean.

Proposition 3.7. Let B_i $(i = 1, 2)$ be complete atomic Boolean lattices. Then $B_1 \otimes B_2$ is Boolean.

Proposition 3.8. Let B_i $(i = 1, 2)$ as above, then $B_1 \otimes B_2$ is isomorphic to $B_2 \otimes B_1$. Moreover, there exists an isomorphism from $B_1 \otimes B_2$ onto $B_2 \otimes B_1$ which is natural with respect to the embeddings of B_1 and B_2 into these lattices, respectively.

The following proposition gives a hint that out construction is in accord with classical point mechanics: the propositional system is the power set $\mathscr{P}(X)$ of some set X, the phase space. Given another system with phase space Y, the propositional system of the compound system is $\mathscr{P}(X \times Y)$.

Proposition 3.9. $\mathscr{P}(X \times Y)$ is isomorphic to $\mathscr{P}(X) \otimes \mathscr{P}(Y)$.

4. *Measurement Processes*

We now continue the discussion that we began in the second part of Section 2, keeping the notation introduced there. L denotes the propositional system of a quantum object and B, which is Boolean, the propositional system of the apparatus. The coupling between both is assumed to be the quasi-tensor-product and is denoted by T.

We first assume that L has trivial center and show by the following proposition that there does not exist an apparatus with Boolean propositional system which defines a nontrivial or nonabsurd proposition in common with its quantal negation. Stated with the notation of PSM I and PSM II, such purely classical apparatus cannot define a question α such that $\varphi[\alpha] = [\nu\alpha]$, where $[\alpha]$ is the proposition holding true for all ensembles of objects for which the outcome b_{out} always occurs, and $[\nu\alpha]$ is the proposition holding true for all ensembles for which the outcome never occurs. Stated in the language of Ludwig (1970), this result is that there is no purely classical apparatus that defines a decision effect.

Since the structure that enters into the description of physical objects by propositional systems is rather elementary, this proves by very general arguments that useful measurement apparatuses, which decide sharply between the truth of a quantal proposition and the truth of its quantal negation, must have a quantal microstructure behind their classical behavior with respect to macroscopic observation.

Proposition 4.1. Let L have trivial center, $a_0, a_1 \in L$, $b_{\text{in}}, b_{\text{out}} \in B$, $b_{\text{in}} \neq \varnothing$, $b_{\text{out}} \neq \varnothing$. Moreover, let U be an automorphism of T, and let the relatons (j) and (jj) of Section 2 hold. Then $a_0 = \varphi a_1$ implies $a_1 \in \{\varnothing, I\}$ in L.

Proof. By Corollary 3.3 $\theta_B(B)$ coincides with the center of T. Hence $\theta_B(b_{\text{in}})$, $\theta_B(b_{\text{out}})$, and $m = U^{-1} \circ \theta_B(b_{\text{out}})$ are in the center of T. Since $a_0 = \varphi a_1$ is assumed, we have equalities (ii) and (iii) of Section 2, which are

$$\theta_L(a_1) \wedge \theta_B(b_{\text{in}}) = m \wedge \theta_B(b_{\text{in}})$$
$$\theta_L(a_0) \wedge \theta_B(b_{\text{in}}) = \psi m \wedge \theta_B(b_{\text{in}})$$

The left-hand sides are in the center of T, too. So there are suitable elements $d_1, d_0 \in B$ such that

$$\theta_B(d_1) = \theta_L(a_1) \wedge \theta_B(b_{\text{in}})$$
$$\theta_B(d_0) = \theta_L(a_0) \wedge \theta_B(b_{\text{in}})$$

from which we conclude

$$\theta_B(d_1) \leqslant \theta_L(a_1), \qquad \theta_B(d_0) \leqslant \theta_L(a_0)$$

Since

$$\theta_B(d_1) \vee \theta_B(d_0) = [m \wedge \theta_B(b_{\text{in}})] \vee [\psi m \wedge \theta_B(b_{\text{in}})] = \theta_B(b_{\text{in}}) \neq \varnothing$$

$\theta_B(d_1) \neq \varnothing$ or $\theta_B(d_0) \neq \varnothing$. By Postulate 5.2 the first case leads to $\theta_L(a_1) = I$, and the second to $\theta_L(a_0) = I$. Since $a_0 = \varphi a_1$, the statement is proved. □

We now assume L to have nontrivial center Z and generalize Proposition 4.1 showing that all elements of the center of L are commensurable. The apparatus, which we will construct, has Boolean propositional system B.

We take B to be isomorphic to Z and identify both. This is possible since Z is a complete atomic Boolean lattice. Then L can be represented as

$$L = \bigoplus_{e \in \mathscr{A}(Z)} [\varnothing, e]$$

where $\mathscr{A}(Z)$ is the set of atoms of Z and square brackets denote formation of segments. We introduce for $e \in \mathscr{A}(Z)$

$$\P(e): L \to [\varnothing, e]$$
$$a \mapsto e \wedge a$$

Then we have for $f \in Z \otimes L$ the formula

$$f(e) = \bigvee_{\tilde{e} \in \mathscr{A}(Z)} \P(\tilde{e}) \circ f(e), \qquad e \in \mathscr{A}(Z)$$

which we will use freely in the following. Constructing a measurement process in $Z \otimes L$ showing the elements of Z to be commensurable, we have to fix an element $b_{\text{in}} \in Z$, an isomorphism $b_{\text{out}}: Z \to Z$, and an automorphism $U: Z \otimes L \to Z \otimes L$, such that for any $z \in Z$

$$\tau(z) \wedge \sigma(b_{\text{in}}) = m(z) \wedge \sigma(b_{\text{in}})$$

and

$$\tau(\varphi(z)) \wedge \sigma(b_{\text{in}}) = \psi m(z) \wedge \sigma(b_{\text{in}})$$

hold, where $m(z) := U^{-1} \circ \sigma \circ b_{\text{out}}(z)$. The second equation can be dropped, since the first will be proved for all $z \in Z$ and b_{out} being an isomorphism. Then detecting the proposition $z \in Z$ on the object system means that $b_{\text{out}}(z)$ has occurred on the apparatus.

We first construct the mapping $U^{-1} =: V$. In order to do this, we make use of the Löwenheim-Skolem-Tarski theorem (e.g., Grätzer, 1968): Any set with cardinal number $\mathscr{K}$ is in bijection with some group of order $\mathscr{K}$. Hence, we identify $\mathscr{A}(Z)$ with a suitable group of cardinality $|\mathscr{A}(Z)|$ and denote the group composition of $e_0, e_1 \in \mathscr{A}(Z)$ by $e_0 \cdot e$.

Lemma 4.2. The mapping

$$V: Z \otimes L \to Z \otimes L$$

defined by

$$(Vf)(e) = \bigvee_{\tilde{e} \cdot \in \mathscr{A}(Z)} ¶(e \cdot \tilde{e}) \circ f(\tilde{e})$$

$f \in Z \otimes L$, $e \in \mathscr{A}(Z)$, is an automorphism.

Proof. We prove injectivity. For any $f \in Z \otimes L$ and arbitrary $e_0, e_1 \in \mathscr{A}(Z)$ we have

$$(Vf)(e_0 \cdot e_1^{-1}) = \bigvee_{\tilde{e} \in \mathscr{A}(Z)} ¶(e_0 \cdot e_1^{-1} \cdot \tilde{e}) \circ f(\tilde{e})$$

hence the formula

$$¶(e_0)\,[Vf(e_0 \cdot e_1^{-1})] = ¶(e_0)\,[f(e_1)]$$

Since for $g \in Z \otimes L$ $f \neq g$ holds if and only if $f(e_1) \neq g(e_1)$ for some $e_1 \in \mathscr{A}(Z)$, and the latter being equivalent to $¶(e_0)[f(e_1)] \neq ¶(e_0)[g(e_1)]$, for some $e_0 \in \mathscr{A}(Z)$, injectivity is easily derived from the above formula.

We prove bijectivity for the restriction $V|_{\mathscr{A}(Z \otimes L)}: \mathscr{A}(Z \otimes L) \to \mathscr{A}(Z \otimes L)$. Recall that these atoms are given by the "characteristic" functions f_{e_0, ϵ_0}, $e_0 \in \mathscr{A}(Z)$, $\epsilon_0 \in \mathscr{A}(L)$ (cf. Section 3).

Let $e_1, \hat{e} \in \mathscr{A}(Z)$, $\epsilon_0 \leqslant e_1$,[3] and $\hat{e} = e_1 \cdot e_0^{-1}$. Then, for $e \in \mathscr{A}(Z)$

$$\begin{aligned}(Vf_{e_0,\epsilon_0})(e) &= \bigvee_{\tilde{e} \in \mathscr{A}(Z)} ¶(e \cdot \tilde{e}) \circ f_{e_0,\epsilon_0}(\tilde{e}) \\ &= ¶(e \cdot e_0)\epsilon_0 = f_{\hat{e},\epsilon_0}(e)\end{aligned}$$

[3] Notice that atoms in Z are not necessarily atoms in L.

since $\P(e \cdot e_0)(\epsilon_0) \neq \varnothing$ if and only if $e \cdot e_0 = e_1$, which implies $e = \hat{e}$. Let there now be given $\hat{e}$, and ϵ_0 arbitrarily. Since we can find $e_0 = \hat{e}^{-1} \cdot e_1$, where e_1 is the unique $e_1 \in \mathscr{A}(Z)$ with $\epsilon_0 \leqslant e_1$, surjectivity of V on the atoms is proven. Since V is injective, it is bijective.

By straightforward computation one shows V to be natural with respect to arbitrary meets and joins. Hence, V is a bijective lattice morphism.

We show naturality of V with respect to the orthocomplementation, if V is applied to atoms of $Z \otimes L$. We have for $e_0, e \in \mathscr{A}(Z)$, $\epsilon_0 \in \mathscr{A}(L)$

$$\begin{aligned} V \circ \psi(f_{e_0,\epsilon_0})(e) &= \bigvee_{\tilde{e} \in \mathscr{A}(Z)} \P(e \cdot \tilde{e}) \circ \varphi f_{e_0,\epsilon_0}(\tilde{e}) \\ &= \left(\bigvee_{\tilde{e} \in \mathscr{A}(Z) \setminus \{e_0\}} e \cdot \tilde{e} \right) \vee [(e \cdot e_0) \wedge \varphi\epsilon_0] \\ &= \left(\bigvee_{\tilde{e} \in \mathscr{A}(Z)} e \cdot \tilde{e} \right) \wedge \left\{ \bigvee_{\tilde{e} \in \mathscr{A}(Z) \setminus \{e_0\}} [(e \cdot \tilde{e}) \vee \varphi\epsilon_0)] \right\} \\ &= \bigvee_{\tilde{e} \in \mathscr{A}(Z) \setminus \{e_0\}} \varphi(\varphi(e \cdot \tilde{e}) \wedge \epsilon_0) \\ &= \varphi \bigwedge_{\tilde{e} \in \mathscr{A}(Z) \setminus \{e_0\}} [\varphi(e \cdot \tilde{e}) \wedge \epsilon_0] \end{aligned}$$

Now let e_1 be the unique element of $\mathscr{A}(Z)$ with $e_1 \geqslant \epsilon_0$, i.e. $\epsilon_0 \wedge \varphi(e_1) = \varnothing$ and $\epsilon_0 \wedge \varphi(\tilde{\tilde{e}}) = \epsilon_0$, $\tilde{\tilde{e}} \neq e_1$, $\tilde{\tilde{e}} \in \mathscr{A}(Z)$. Hence

$$\bigwedge_{\tilde{e} \in \mathscr{A}(Z) \setminus \{e_0\}} [\varphi(e \cdot \tilde{e}) \wedge \epsilon_0] = \begin{cases} \epsilon_0 & \text{if } e \cdot e_0 = e_1 \\ \varnothing & \text{otherwise} \end{cases}$$

Let $\hat{e} := e_1 \cdot e_0^{-1}$; then we have

$$V(f_{e_0,\epsilon_0})(e) = f_{\hat{e},\epsilon_0}(e) = \bigwedge_{\tilde{e} \in \mathscr{A}(Z) \setminus \{e_0\}} [\varphi(e \cdot \tilde{e}) \wedge \epsilon_0]$$

Hence

$$V \circ \psi(f_{e_0,\epsilon_0}) = \psi \circ V(f_{e_0,\epsilon_0})$$

Summing up, we have found V to be bijective and natural with respect to ψ on the atoms of $Z \otimes L$, and, V is natural with respect to arbitrary meets and joins. Since $Z \otimes L$ is complete and atomic, we have found that V is an isomorphism. □

We now put $b_{\text{in}} = e_0$, $e_0 \in \mathscr{A}(Z)$. The following lemma states that there is a unique isomorphism $b_{\text{out}}: Z \to Z$, such that, taking $U^{-1} = V$, in $\mathscr{M}(Z, L)$ for any $z \in Z$ the equation

$$\tau(z) \wedge \sigma(e_0) = [U^{-1} \circ \sigma \circ b_{\text{out}}(z)] \wedge \delta(e_0) \qquad (*)$$

holds true. σ and τ denote the embeddings of Z and L in $\mathscr{M}(Z, L)$. Hence, relations (i)–(iii) of Section 2 hold for any $z \in Z$.

Lemma 4.3. The mapping $\hat{e} \to e_0^{-1} \cdot \hat{e}$, $e \in \mathscr{A}(Z)$, induces a unique automorphism of Z. Denote this automorphism by b_{out}; then equation (*) holds true. Conversely, b_{out} is uniquely determined by (*).

Proof. The mapping $\hat{e} \to e_0^{-1} \cdot \hat{e}$ is a bijection of $\mathscr{A}(Z)$, hence it induces an isomorphism on Z.

For the remainder of proof we can restrict ourselves on the case $z \in \mathscr{A}(Z)$. For $z \in \mathscr{A}(Z)$ we have

$$
\begin{aligned}
&[V \circ \sigma(e_0^{-1} \cdot \hat{e})]\,(e) \wedge \sigma(e_0)(e) \\
&\qquad = \left\{ \bigvee_{\tilde{e} \in \mathscr{A}(Z)} [(e \cdot \tilde{e}) \wedge \sigma(e_0^{-1} \cdot \hat{e})(\tilde{e})] \right\} \wedge \sigma(e_0)(e) \\
&\qquad = [(e \cdot e_0^{-1} \cdot \hat{e}) \wedge I] \wedge \sigma(e_0)(e) \\
&\qquad = \hat{e} \wedge \sigma(e_0)(e) = \tau(\hat{e})(e) \wedge \sigma(e_0)(e)
\end{aligned}
$$

That b_{out} is uniquely determined by (*) is easy to check. □

Summing up Lemma 4.2 and Lemma 4.3, we have shown the following.

Theorem 4.4. Let L be the propositional system of some quantum object. The elements of the center Z of L are commensurable propositions and can be measured together by an apparatus with Boolean propositional system $B = Z$.

A trivial byproduct of this theorem is that all propositions of a classical system are commensurable by an apparatus with Boolean propositional system.

Acknowledgment

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Appendix

We will give a proof of Lemma 2 of PSM II. Before doing so, we propose some statements that will be used.

Proposition A.1. Let B_1 and B_2 be Boolean sublattices of a complete atomic orthomodular lattice T, and let $B_1 \leftrightarrow B_2$, i.e., the elements of $B_1 \cup B_2$ are mutually compatible. Then there is a Boolean sublattice B of T such that $B_1 \subseteq B$ and $B_2 \subseteq B$.

Proof. Let

$$
M := \{m \in T \mid m = \bigwedge_{r \in \mathscr{R}} (b_r^{(1)} \vee b_r^{(2)}),\ b_r^{(1)} \in B_1, b_r^{(2)} \in B_2,
$$

$\mathscr{R}$ being a finite indexing set

Obviously, $B_1 \subseteq M$ and $B_2 \subseteq M$. Moreover, all elements of M are mutually compatible since $a, a_k \in T$, $k \in \mathscr{K}$, $\mathscr{K}$ finite indexing set, $a \leftrightarrow a_k$ implies $a \leftrightarrow \bigwedge_{k\in\mathscr{K}} a_k$ and $a \leftrightarrow \bigvee_{k\in\mathscr{K}} a_k$ (e.g., C. Piron, 1964, Theorem IX).

M is closed with respect to the orthocomplementation. We give the proof by induction. We have for $b^{(1)} \in B_1, b^{(2)} \in B_2$ $\psi(b^{(1)} \vee b^{(2)}) = (\psi b^{(1)} \wedge \psi b^{(2)}) = (\psi b^{(1)} \vee \varnothing) \wedge (\varnothing \vee \psi b^{(2)})$. Now let $b_n^{(1)} \in B_1$ and $b_n^{(2)} \in B_2$, $n = 1, 2, 3, \ldots, n_0 + 1$, and assume the statement to hold for $n \leqslant n_0$. Then we have

$$\psi\left(\bigwedge_{n=1,2,\ldots,n_0+1} (b_n^{(1)} \vee b_n^{(2)})\right) = \left[\bigwedge_{j\in\mathscr{F}} (\tilde{b}_j^{(1)} \vee \tilde{b}_j^{(2)})\right] \vee (\psi b_{n_0+1}^{(1)} \wedge \psi b_{n_0+1}^{(2)})$$

where $b_j^{(1)} \in B_1, b_j^{(2)} \in B_2$ and $\mathscr{F}$ is a finite indexing set. Hence

$$\psi\left(\bigwedge_{n=1,\ldots,n_0+1} (b_n^{(1)} \vee b_n^{(2)})\right) = \bigwedge_{j\in\mathscr{F}} [(\tilde{b}_j^{(1)} \vee \tilde{b}_j^{(2)}) \vee (\psi b_{n_0+1}^{(1)} \wedge \psi b_{n_0+1}^{(2)})]$$

$$= \bigwedge_{j\in\mathscr{F}} \{[(\tilde{b}_j^{(1)} \vee \psi b_{n_0+1}^{(1)}) \vee \tilde{b}_j^{(2)}] \wedge [\tilde{b}_j^{(1)} \vee (\tilde{b}_j^{(2)} \vee \psi b_{n_0+1}^{(2)})]\}$$

which has the desired form and is, hence, in M.

Since M is obviously closed forming finite meets, it follows that it is also closed forming finite joins. Hence M is a Boolean sublattice of T containing B_1 and B_2 as sublattices. □

Corollary. M is the Boolean sublattice of T, generated by $B_1 \cup B_2$.

The proof of the corollary is obvious and will be omitted. By a simple Zorn lemma argument, which we also omit, one proves the following proposition.

> *Proposition A.2.* Let T be as above. For any $a \in T$ there is a maximal Boolean sublattice $B_a \subseteq T$ such that $a \in B_a$.

We remark, that any maximal Boolean sublattice of T is complete (e.g., Piron, 1964, Theorem X).

We now recall some notation and results of McLaren (1964) which we will use in the proof of the lemma. Let S be a partially ordered orthocomplemented set. We maintain our notation ψ for the orthocomplementation. For $a, b \in S$ let $a \perp b$ be defined by $a \leqslant \psi(b)$. For any subset $A \subseteq S$ let $A^\perp := \{s \in S | s \perp a \text{ for any } a \in A\}$, and $A^- := A^{\perp\perp}$. The system $L(S) := \{A \subseteq S | A = A^-\}$ of closed subsets of S is partially ordered by inclusion and a complete lattice, for which infima equal to set theoretical meets. $L(S)$ is orthocomplemented by the mapping $A \to A^\perp$. The mapping $S \to L(S)$, $a \to \{a\}^-$ is an embedding. Moreover, $L(S)$ can be identified with the completion by cuts of S.

Note that for one-elementary subsets of S the following formula holds:

$$\{s\}^- = \{a \in S | a \leqslant s\}$$

Let there now be given two complete atomic orthomodular lattices L_1 and L_2. In Postulates 5.1–5.3 replace B by L_1 and L by L_2, and assume these postulates to hold for a given complete atomic orthomodular lattice T. The embeddings of L_1 and L_2 into T will be denoted by θ_1 and θ_2, respectively.

> *Lemma 2 of PSM II.* Let $\epsilon \in \mathscr{A}(L_1)$ and $e \in \mathscr{A}(L_2)$. Then $\theta_1(\epsilon) \wedge \theta_2(e)$ is an atom of T.

Proof. Let B_ϵ and B_e be maximal Boolean sublattices of L_1 and L_2, respectively, such that $\epsilon \in B_\epsilon$ and $e \in B_e$. Then, for any $b_\epsilon \in B_\epsilon$, $b_\epsilon \neq \varnothing$, $b_\epsilon \neq I$, we have either $\epsilon \leqslant b_\epsilon$, or $\epsilon \leqslant \varphi b_\epsilon$, the latter being equivalent to $b_\epsilon \leqslant \varphi\epsilon$. Analogously, we have for $b_e \in B_e$, $b_e \neq \varnothing$, $b_e \neq I$, either $e \leqslant b_e$ or $b_e \leqslant \varphi e$.

Let M denote the Boolean sublattice generated by $\theta_1(B_\epsilon) \cup \theta_2(B_e)$ in T. From the proof of Proposition A.1 we know that any element $m \in M$ can be written in the form

$$m = \bigwedge_{r \in \mathscr{R}} [\theta_1(b_r^{(\epsilon)}) \vee \theta_2(b_r^{(e)})]$$

where $b_r^{(\epsilon)} \in B_\epsilon$ and $b_r^{(e)} \in B_e$, $\mathscr{R}$ being a finite indexing set. Let $m \neq \varnothing$, and $m \neq I$. Without restriction of generality we can then assume that $b_r^{(\epsilon)} \neq I$ and $b_r^{(e)} \neq I$, $r \in \mathscr{R}$, since the respective terms do not contribute to m. Moreover, the case that $b_r^{(\epsilon)} = \varnothing$ and $b_r^{(e)} = \varnothing$ is excluded, since then $m = \varnothing$ would hold. We show that for each $r \in \mathscr{R}$, separately, one of the two alternatives

$$\theta_1(b_r^{(\epsilon)}) \vee \theta_2(b_r^{(e)}) \leqslant \theta_1(\varphi\epsilon) \vee \theta_2(\varphi e)$$

or

$$\theta_1(b_r^{(\epsilon)}) \vee \theta_2(b_r^{(e)}) \geqslant \theta_1(\epsilon) \wedge \theta_2(e)$$

holds. The first alternative arises if both $b_r^{(\epsilon)} \leqslant \varphi\epsilon$ and $b_r^{(e)} \leqslant \varphi e$ hold true. The second arises otherwise. Consider now $m \in M$, $m \neq \varnothing$, $m \neq I$ and assume a representation of the above form. If the first alternative arises for some $r \in \mathscr{R}$, we have

$$m = \bigwedge_{r \in \mathscr{R}} [\theta_1(b_r^{(\epsilon)}) \vee \theta_2(b_r^{(e)})] \leqslant \theta_1(\varphi\epsilon) \vee \theta_2(\varphi e)$$

If the first alternative arises for no $r \in \mathscr{R}$, we have

$$m = \bigwedge_{r \in \mathscr{R}} [\theta_1(b_r^{(\epsilon)}) \vee \theta_2(b_r^{(e)})] \geqslant \theta_1(\epsilon) \wedge \theta_2(e)$$

Thus we have proven that in M there are no nonabsurd elements that precede $\theta_1(\epsilon) \wedge \theta_2(e)$.

We are now going to show the latter statement to hold also for the completion of M by cuts. To that end let $\tilde{m} \in L(M)$, $\tilde{m} \neq \{\varnothing\}^-$ and assume that there is no element $m_1 \in \tilde{m}$ such that $m_1 \geqslant \theta_1(\epsilon) \wedge \theta_2(e)$. Then $m_1 \leqslant \theta_1(\varphi\epsilon) \vee \theta_2(\varphi e)$ for any $\tilde{m}_1 \in m$. Hence, we have

$$\tilde{m} \leqslant \{\theta_1(\varphi e) \vee \theta_2(\varphi e)\}^-$$

Assume now, on the contrary, that there is an element $m_2 \in \tilde{m}$ such that $m_2 \geqslant \theta_1(\epsilon) \wedge \theta_2(e)$. Then

$$\{\theta_1(\epsilon) \wedge \theta_2(e)\}^- \leqslant \{m_2\}^- \leqslant \tilde{m}$$

Identifying now $L(M)$ with the completion by cuts of M, we have for $\tilde{m} \neq \{\varnothing\}^-$ either

$$\tilde{m} \leqslant \psi(\theta_1(\epsilon) \wedge \theta_2(e)) \qquad \text{or } \tilde{m} \geqslant \theta_1(\epsilon) \wedge \theta_2(e)$$

Since the completion by cuts of M is postulated to be maximal Boolean in T, atoms in $L(M)$ must also be atoms in T. This completes the proof of Lemma 2 of PSM II.

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